Evolution of cosmological dark matter perturbations

Antony Lewis[1,*] and Anthony Challinor[2,†]
[1]DAMTP, CMS, Wilberforce Road, Cambridge CB3 0WA, UK.
[2]Astrophysics Group, Cavendish Laboratory, Madingley Road, Cambridge CB3 OHE, UK.

We discuss the propagation of dark matter perturbations with non-zero velocity dispersion in cosmological models. In particular a non-zero massive neutrino component may well have a significant effect on the matter power spectrum and cosmic microwave background anisotropy. We present a covariant analysis of the evolution of a dark matter distribution via a two-dimensional momentum-integrated hierarchy of multipole equations. This can be expanded in the velocity weight to provide accurate approximate equations if the matter is non-relativistic, and we also perform an expansion in the mass to study the propagation of relativistic matter perturbations. We suggest an approximation to the exact hierarchy that can be used to calculate efficiently the effect of the massive neutrinos on the CMB power spectra. We implement the corresponding scalar mode equations numerically achieving a considerable reduction in computation time compared with previous approaches.

I. INTRODUCTION

Current observational evidence suggests that a significant fraction of the matter density of the universe is in the form of dark matter. Most of this must have low velocity dispersion to be consistent with large scale structure, and the cold dark matter (CDM) model has so far proved to be a good model. However it is quite possible that the dark matter has non-zero velocity dispersion, in the form of warm dark matter. There is also good evidence that the neutrinos have non-zero mass, so there will be a small component of hot dark matter in the form of massive neutrinos. In this paper we study the evolution of dark matter perturbations with a non-zero velocity dispersion, focusing especially on the massive neutrinos and their effect on the Cosmic Microwave Background (CMB) anisotropy.

One of the useful products of forthcoming high-resolution CMB observations should be good limits on the massive neutrino masses. Atmospheric neutrino oscillation observations currently provide three-sigma evidence for a mass difference $\Delta m^2 \sim (3 \pm 2) \times 10^{-3}\,\mathrm{eV}^2$ [1]. This is consistent with a predominantly ν_μ-ν_τ oscillation though more complicated interpretations are possible. Solar neutrino data show a much smaller mass difference $\Delta m^2 \sim (3 \pm 1) \times 10^{-4}\,\mathrm{eV}^2$ [2], suggesting that two of the neutrino mass eigenstates are likely to be nearly degenerate or very small. The neutrino mass is related to Ω_ν, the ratio of the density of massive neutrinos to the critical density required for a flat universe, by (see e.g. Ref. [3])

$$\Omega_\nu h^2 = \frac{\sum m_\nu}{93.8\,\mathrm{eV}}. \tag{1}$$

For $h \sim 0.7$ the neutrino oscillations therefore provide the bound that $\Omega_\nu \gtrsim 10^{-3}$. Recently there has also been controversial evidence for double beta decay [2, 4], which could place interesting lower limits on the neutrino mass eigenstates if confirmed. Current cosmological evidence is not very constraining, suggesting only that $\sum m_\nu \lesssim 4.2\,\mathrm{eV}$ and hence $\Omega_\nu \lesssim 0.1$ [5].

If $\Omega_\nu \sim 10^{-3}$ the observable effect on the cosmology is limited, with the CMB temperature and polarization power spectra changing by less than a percent relative to the corresponding pure CDM model. However at slightly larger values the effects can become important. For $\Omega_\nu \sim 3 \times 10^{-3}$ there are changes in the electric polarization power spectrum at the few percent level. For $\Omega_\nu \gtrsim 5 \times 10^{-3}$ the effect on the temperature power spectrum becomes significant at the percent level. Though it might be more natural for all the neutrino masses to be very small, three neutrinos of approximately degenerate mass ($\Delta m < 0.07\,\mathrm{eV}$) would be consistent with current data and could easily be cosmologically interesting.

Even if the contribution is small, it is essential to be able to include the effect of massive neutrinos in the theoretical calculations, both to constrain the neutrino masses themselves, and for accurate recovery of the other cosmological

*Electronic address: Antony@AntonyLewis.com
†Electronic address: A.D.Challinor@mrao.cam.ac.uk

parameters. Even if neutrino experiments suggest that the masses are very small we would also like to check independently that this is consistent with cosmology.

In this paper we analyse the propagation of dark matter perturbations, presenting evolution equations that can be used for fast numerical implementations. We shall focus on linearized massive neutrino perturbations, though our results could be applied equally well to other forms of dark or interacting non-massless matter. In particular the low-energy equations could be used for the efficient propagation of warm dark matter perturbations. We also provide exact equations that may be a useful starting point for studying non-linear evolution.

We use a covariant expansion of the dark matter distribution function following Ref. [6], giving a set of exact multipole equations for the propagation of collisionless matter. To compute the matter power spectrum and CMB anisotropies in models with hot or warm dark matter we linearize the multipole equations about an exact Friedmann-Robertson-Walker (FRW) model. The covariant analysis yields sets of physically transparent equations that can be split into scalar, vector and tensor parts as required for numerical implementation (for previous applications of this approach see Refs. [6, 7, 8, 9]). We derive useful limiting cases when the matter is highly relativistic and non-relativistic, and also discuss an accurate approximate scheme that interpolates between the two regimes. Though we present a covariant analysis here, our approximation techniques could be applied equally well using metric variables, for example the synchronous or Newtonian gauge formalism of Ref. [10], or the 'gauge-ready' formalism of Ref. [11].

For parameter estimation and hypothesis testing with forthcoming CMB data a large number of accurate theoretical power spectra will need to be generated, even if fast Monte Carlo sampling techniques are employed. This can be very time consuming even using efficient codes such as CMBFAST [12] or parallelized derivatives such as CAMB [13]. Using current methods the massive neutrino evolution dominates the computation time, so our methods for speeding up the neutrino evolution should be useful. Our numerical code can generate CMB power spectra (including polarization) two or three times faster than CMBFAST on one processor, and in under a second on a modern 16 processor machine (non-flat models take about twice as long).

We employ general relativity with signature $(+---)$ and use natural units where $c=1$.

II. COVARIANT MULTIPOLE EQUATIONS

We analyse the propagation of the dark matter with respect to a reference 4-velocity u^a. The variables we use are then physically meaningful and can in principle be measured directly by an observer moving with this velocity. We decompose the 4-momentum of a particle of mass m with respect to u^a as

$$p^a = Eu^a + \lambda e^a, \tag{2}$$

where E is the energy, $\lambda = \sqrt{E^2 - m^2}$ is the 3-momentum and $e^a e_a = -1$. We shall assume that the dark matter is collisionless so that the distribution function $f = f(x^a, p^a)$ obeys the Liouville equation

$$\partial_\tau f = 0 \tag{3}$$

along a path described by an affine parameter τ (defined so that $\partial_\tau x^a = p^a$).

The e^a dependence of the distribution function $f = f(x, E, e^a)$ can be expanded in terms of irreducible multipoles as

$$f = \sum_{l=0}^{\infty} F_{A_l} e^{A_l} = F + F_a e^a + F_{ab} e^a e^b + \ldots, \tag{4}$$

where the tensors $F_{A_l} = F_{\langle a_1 \ldots a_l \rangle}$ are projected (orthogonal to u^a) symmetric and trace-free. The angle brackets denote the projected symmetric trace free (PSTF) part of the enclosed indices. This multipole expansion is the covariant equivalent of the usual spherical harmonic expansion. Substituting into the Liouville equation we have

$$\sum_{l=0}^{\infty} \left(\partial_E F_{A_l} \partial_\tau E \, e^{A_l} + p^a \nabla_a F_{A_l} e^{A_l} + l F_{A_l} p^a \nabla_a e^{a_l} e^{A_{l-1}} \right) = 0. \tag{5}$$

The propagation is governed by the geodesic equation $p^a \nabla_a p^b = 0$, and the component in the u^a direction gives

$$(E/\lambda)\partial_\tau E = \partial_\tau \lambda = E^2 A_a e^a + E\lambda(\sigma_{ab} e^a e^b - H). \tag{6}$$

This then implies that

$$h^a{}_b p^c \nabla_c e^b = -\tfrac{E^2}{\lambda}(A^a + e^a A^c e^a) - E(e^b \omega_b{}^a + \sigma_{bc} e^b e^c e^a + \sigma^a{}_b e^b), \tag{7}$$

where h_{ab} is the projection tensor into the space orthogonal to u^a. Here, the local Hubble parameter, acceleration, shear and vorticity are given by

$$H = \tfrac{1}{3}\nabla^a u_a, \qquad A_a = \dot{u}_a \equiv u^b \nabla_b u_a, \qquad \sigma_{ab} = \mathrm{D}_{\langle a} u_{b\rangle}, \qquad \omega_{ab} = \mathrm{D}_{[a} u_{b]}, \tag{8}$$

and $\mathrm{D}_a X_{a_1 \ldots a_l} = h_a{}^b h_{a_1}{}^{b_1} \ldots h_{a_l}{}^{b_l} \nabla_b X_{b_1 \ldots b_l}$ is the spatial covariant derivative. Using these relations in the Liouville equation and equating irreducible PSTF terms one obtains a set of equations for the evolution of the distribution function multipoles:

$$\begin{aligned}
E\, {}_\perp\dot{F}_{A_l} - \ & \lambda^2 H \partial_E F_{A_l} - \tfrac{l+1}{2l+3}\lambda \mathrm{D}^a F_{aA_l} + \lambda \mathrm{D}_{\langle a_l} F_{A_{l-1}\rangle} - lEF_{a\langle A_{l-1}}\omega_{a_l\rangle}{}^a \\
& + \left[\lambda E \partial_E F_{\langle A_{l-1}} - (l-1)\tfrac{E^2}{\lambda} F_{\langle A_{l-1}}\right] A_{a_l\rangle} - \tfrac{l+1}{2l+3}\left[(l+2)\tfrac{E^2}{\lambda}F_{aA_l} + \lambda E \partial_E F_{aA_l}\right] A^a \\
& - \tfrac{l}{2l+3}\left[3EF_{a\langle A_{l-1}} + 2\lambda^2 \partial_E F_{a\langle A_{l-1}}\right]\sigma_{a_l\rangle}{}^a + \tfrac{(l+1)(l+2)}{(2l+3)(2l+5)}\left[(l+3)EF_{abA_l} + \lambda^2 \partial_E F_{abA_l}\right]\sigma^{ab} \\
& + \left[\lambda^2 \partial_E F_{\langle A_{l-2}} - (l-2)EF_{\langle A_{l-2}}\right]\sigma_{a_{l-1}a_l\rangle} = 0,
\end{aligned} \tag{9}$$

where ${}_\perp\dot{F}_{A_l} = h^{B_l}{}_{A_l} u^a \nabla_a F_{B_l}$ is the projected time derivative.

The dark matter couples to other matter via gravity according to general relativity. The relevant quantities for the gravitational interaction are not the distribution function multipoles but rather the integrated quantities that make up the stress-energy tensor. Following Ref. [6] we therefore integrate over momentum defining the momentum-integrated multipoles

$$J^{(i)}_{A_l} \equiv \frac{4\pi(-2)^l (l!)^2}{(2l+1)!} \int_0^\infty \mathrm{d}\lambda\, \lambda^2 E F_{A_l} \left(\frac{\lambda}{E}\right)^n, \tag{10}$$

where $n \equiv l + 2i$ is the velocity weight. The numerical factor is introduced so that the energy density, momentum density, anisotropic stress, and isotropic pressure that make up the stress-energy tensor are simply

$$\rho = J^{(0)}, \qquad q_a = J^{(0)}_a, \qquad \pi_{ab} = J^{(0)}_{ab}, \qquad p = \tfrac{1}{3}J^{(1)}. \tag{11}$$

Integrating Eq. (9) over momentum we obtain the set of exact multipole equations

$$\begin{aligned}
{}_\perp\dot{J}^{(i)}_{A_l} & + H\left[(3+n)J^{(i)}_{A_l} + (1-n)J^{(i+1)}_{A_l}\right] + \mathrm{D}^a J^{(i)}_{aA_l} - \tfrac{l}{2l+1}\mathrm{D}_{\langle a_l} J^{(i+1)}_{A_{l-1}\rangle} - lJ^{(i)}_{a\langle A_{l-1}}\omega_{a_l\rangle}{}^a \\
& + \tfrac{l}{2l+1}\left[(l+n+1)J^{(i)}_{\langle A_{l-1}} + (2-n)J^{(i+1)}_{\langle A_{l-1}}\right] A_{a_l\rangle} + \left[(l-n)J^{(i-1)}_{aA_l} + (n-2)J^{(i)}_{aA_l}\right] A^a \\
& + \tfrac{l}{2l+3}\left[(2n+3)J^{(i)}_{a\langle A_{l-1}} + (2-2n)J^{(i+1)}_{a\langle A_{l-1}}\right]\sigma_{a_l\rangle}{}^a + \left[(l-n)J^{(i-1)}_{abA_l} + (n-1)J^{(i)}_{abA_l}\right]\sigma^{ab} \\
& + \tfrac{l(l-1)}{4l^2-1}\left[(n-1)J^{(i+2)}_{\langle A_{l-2}} - (l+n+1)J^{(i+1)}_{\langle A_{l-2}}\right]\sigma_{a_{l-1}a_l\rangle} = 0.
\end{aligned} \tag{12}$$

This exact two-dimensional infinite hierarchy of covariant equations can be used as a starting point for analysing the propagation of dark matter with arbitrary velocity dispersion. In the case of collisional matter the right hand side would contain a collision term. The (l, i) element of the hierarchy involves moments with l in the range $(l-2, l+2)$ and i in the range $(i-1, i+2)$. The coefficients in front of the $i-1$ moments vanish for $i = 0$, so the subset of equations with $i \geq 0$ form a closed system for the $i \geq 0$ moments.

III. COSMOLOGICAL DARK MATTER

The application of the two-dimensional multipole hierarchy we consider here is to dark matter perturbations in cosmology. The universe is assumed to be nearly FRW and we linearize by dropping products of any quantities that vanish in an exact FRW universe. All the $l > 0$ multipoles vanish by isotropy in an exact FRW model and are therefore first order. The only zero-order quantities are the scale factor, Hubble parameter and the $l = 0$ multipoles. To first order about a FRW universe the momentum-integrated multipole equations (12) become

$$\begin{aligned}
\dot{J}^{(i)}_{A_l} & + H\left[(1-n)J^{(i+1)}_{A_l} + (3+n)J^{(i)}_{A_l}\right] - \tfrac{l}{2l+1}\mathrm{D}_{\langle a_l} J^{(i+1)}_{A_{l-1}\rangle} + \mathrm{D}^a J^{(i)}_{aA_l} \\
& + \left[(1+l+n)J^{(i+l-1)} + (3-l-n)J^{(i+l)}\right](\delta_{1l}\,\tfrac{1}{3}A_{a_1} - \delta_{2l}\,\tfrac{2}{15}\sigma_{a_1a_2}) = 0.
\end{aligned} \tag{13}$$

We characterize perturbations to the $J^{(i)}$ (which do not vanish in the FRW limit) by their first-order co-moving spatial gradients

$$\chi_a^{(i)} \equiv S\mathrm{D}_a J^{(i)}. \tag{14}$$

Here, S is the scale factor defined generally in a perturbed universe by integrating $\dot{S} = HS$ with initial conditions chosen to ensure that $\mathrm{D}_a S$ is first order. Taking the spatial gradient of the $l = 0$ equations and commuting derivatives gives the propagation equation

$$\dot{\chi}_a^{(i)} + \left[(1-2i)J^{(i+1)} + (3+2i)J^{(i)}\right]\dot{h}_a + S\mathrm{D}_a\mathrm{D}^b J_b^{(i)} + H\left[(1-2i)\chi_a^{(i+1)} + (3+2i)\chi_a^{(i)}\right] = 0, \tag{15}$$

where $h_a \equiv \mathrm{D}_a S$ is the projected gradient of the scale factor. Note that

$$\dot{h}_a = S(\mathrm{D}_a H - HA_a) \tag{16}$$

is defined uniquely at linear order once a choice is made for u^a, unlike h_a itself which also depends on the choice of surface on which we impose $S =$ constant as an initial condition.

The covariant equations (13) and (15) constitute an infinite two-dimensional hierarchy for analysing the propagation of linear dark matter perturbations. For massless particles $E = \lambda$ and the velocity-weighted moments are identical, $J_{A_l}^{(i)} = J_{A_l}^{(i')}$. The momentum-integrated equations then reduce to the usual one-dimensional Boltzmann hierarchy used to propagate massless neutrino perturbations [7].

For non-relativistic matter the velocity weight $l + 2i$ controls the magnitude of the momentum-integrated moments. If we neglect terms with $l + 2i > n_*$ the order n_* equations become (for $0 \leq 2i < n_* - 2$)

$$\dot{J}_{A_{n_*-2i}}^{(i)} + (3+n_*)HJ_{A_{n_*-2i}}^{(i)} = 0. \tag{17}$$

Using $H = \dot{S}/S$ this solves to give

$$J_{A_{n_*-2i}}^{(i)} \propto S^{-3}S^{-n_*}. \tag{18}$$

As expected the truncated equations will be accurate to within terms that decay as an n_*th-order velocity density. In addition to demanding that $v_* \ll 1$, where v_* is the typical particle velocity, an accurate truncation also requires that the perturbations do not vary too rapidly in space. The spatial derivative terms in the n_*th-order equations can only be neglected if $kv_* \ll SH$, i.e. the free-streaming scale in an expansion time is much less than the proper wavelength S/k of the fluctuations.

A hierarchy truncated at $n_* = 2$ was used in Ref. [14] to study stress effects in dark matter structure formation. For warm or hot dark matter one can choose n_* as large as required to obtain accurate results.

There is a subtlety associated with this low-velocity-weight truncation scheme that arises because the $l = 1$ momentum-integrated moments are not frame-invariant in linear theory. Under linear changes in the fiducial velocity, $u^a \to \tilde{u}^a = u^a + v^a$, one can show that the moments transform as

$$J_{A_l}^{(i)} \to \tilde{J}_{A_l}^{(i)} = J_{A_l}^{(i)} - \tfrac{1}{3}\delta_{l1}v_{a_1}[(3+2i)J^{(i)} + (1-2i)J^{(i+1)}]. \tag{19}$$

For $l = 1$ and $i = 0$ this reduces to the well-known result $\tilde{q}^a = q^a - (\rho + p)v^a$. We see that at $l = 1$ the transformation is not homogeneous in the velocity weight. The form of Eq. (19) implies that the truncation condition $J_{A_l}^{(i)} = 0$ for $l + 2i > n_*$ is only frame-invariant at linear order for n_* odd.

IV. MASSIVE NEUTRINO PERTURBATIONS

We now discuss how to propagate massive neutrino perturbations from the early universe until the present day, with the aim of implementing the equations numerically in an efficient way.

Assuming instantaneous neutrino decoupling in the early universe the zero-order distribution function for massive neutrinos is given by the Fermi-Dirac distribution

$$f(q) \propto \left[\exp\left(\frac{\sqrt{q^2 + m_\nu^2 S_d^2}}{k_B T_d S_d}\right) + 1\right]^{-1}, \tag{20}$$

where T_d and S_d are the temperature and scale factor at neutrino decoupling and $q \equiv S\lambda$ is the co-moving momentum. We assume the particles will be highly relativistic at decoupling so the mass term can be dropped. As the universe expands the momenta redshift and the neutrinos evolve to become non-relativistic.

At hydrogen recombination light neutrinos will still be quite relativistic, and the higher multipoles in the distribution can be significant. Usually it is assumed that the neutrinos become non-relativistic well before recombination, and hence the high multipoles have little effect. For light species it is necessary to evolve the higher multipoles to get accurate results, and approximate schemes like that in Ref. [15], based on a low-l truncation, do not give accurate results. For such relatively light species it is nonetheless still important to take account of the non-zero mass, as this can effect the matter power spectrum and CMB anisotropy power spectrum at the level of several percent (though some of this is trivially accounted for by the change in the background equation of state).

It is usual to assume that if the neutrino mass eigenstates are significant they will be approximately degenerate. This is reasonable as the observed mass squared differences are sufficiently small that taking them into account would make no observable difference to the cosmology for the foreseeable future. However it is possible that at some point in the future observations will be sufficiently accurate that even small mass splittings may have an observable effect, in which case the neutrino mass eigenstates may need to be propagated separately.

The general regime

When the most of the neutrinos are either highly relativistic or highly non-relativistic one can use a small mass or small velocity expansion, as discussed below. However in general a significant fraction of the neutrinos have intermediate velocities and one needs to propagate the distribution function directly to get accurate results. As the neutrinos are locally in equilibrium prior to decoupling, the distribution function evolves from the isotropic form in Eq. (20), but with a spatial variation in the temperature. The efficiency with which free-streaming converts this spatial variation into an angular variation depends on the particle velocity (and the wavelength of the fluctuation). For this reason the anisotropies in the distribution function do not simply inherit the momentum dependence of the spatial variation of the initial (isotropic) distribution, unlike the case for massless particles.

In terms of the co-moving momentum $q \equiv S\lambda$ and energy $\epsilon \equiv SE$ the multipole equations (9) for the propagation of the distribution function linearize to give

$$\dot{F}_{A_l} - \tfrac{q}{\epsilon}\tfrac{l+1}{2l+3}\mathrm{D}^c F_{cA_l} + \tfrac{q}{\epsilon}\mathrm{D}_{\langle a}F_{A_{l-1}\rangle} + \delta_{l1}\left(\tfrac{\epsilon}{q}A_{a_1} + \tfrac{1}{S}\tfrac{q}{\epsilon}h_{a_1}\right)q\partial_q F + \delta_{l2}\sigma_{a_1a_2}q\partial_q F = 0, \tag{21}$$

where the spacetime derivatives are taken at constant q. Note that

$$\nabla_a F_{A_l}|_\lambda = \nabla_a F_{A_l}|_q + (\nabla_a S/S)q\partial_q F_{A_l}. \tag{22}$$

The $l = 1$ equation contains the first order variable combination

$$\mathcal{V}_a \equiv S\mathrm{D}_a F + h_a q\partial_q F = S\mathrm{D}_a F|_\lambda \tag{23}$$

that covariantly characterizes perturbations to the isotropic part of the distribution function. It integrates to give the co-moving gradient of the energy density:

$$\chi_a = \chi_a^{(0)} \equiv S\mathrm{D}_a\rho = \frac{4\pi}{S^4}\int_0^\infty \mathrm{d}q\, q^2\epsilon\mathcal{V}_a. \tag{24}$$

Differentiating Eq. (21) for $l = 0$ with respect to time and commuting derivatives we find the propagation equation

$$\dot{\mathcal{V}}_a = \tfrac{1}{3}\tfrac{q}{\epsilon}S\mathrm{D}_a\mathrm{D}^b F_b + \dot{h}_a q\partial_q F. \tag{25}$$

This equation, together with the $l > 0$ multipole equations, allows one to propagate the distribution function perturbations directly. The synchronous gauge and Newtonian gauge equivalent of these equations are well known [10] and are what is usually used to propagate numerically massive neutrino perturbations [10, 12, 16]. This is slow computationally as one has to propagate a hierarchy of multipoles for each co-moving momentum, and perform numerical integrations to compute the stress-energy tensor that is relevant for coupling to the other perturbations. Even for one massive species the numerical evolution of its distribution function dominates the computation time, and it becomes proportionately worse if one has to evolve several massive species.

Note that when a given co-moving momentum becomes sufficiently non-relativistic ($kq/\epsilon \ll SH$ with k co-moving wavenumber) free-streaming becomes ineffective at converting spatial structure into angular structure, and the spatial-derivative terms in Eq. (21) become negligible. In this limit, $F_{A_l}(q)$ for $l > 2$ is approximately constant. When nearly all the particles are non-relativistic, constancy of $F_{A_l}(q)$ leads directly to the scaling of the momentum-integrated moments given in Eq. (18).

The relativistic regime

In the relativistic regime the bulk of the distribution have momenta large compared to the rest mass $q \gg m_\nu S$. Defining the dimensionless mass parameter $\bar{m} \equiv m_\nu / k_B T_d S_d$ one can expand Eq. (10) in terms of the small quantity $\bar{m}S$ (which is the ratio of mass to the typical proper momentum) giving, in the FRW limit,

$$\rho \approx \rho_0 \left(1 + \frac{5}{7\pi^2}\bar{m}^2 S^2\right), \qquad p \approx \frac{\rho_0}{3}\left(1 - \frac{5}{7\pi^2}\bar{m}^2 S^2\right), \qquad J^{(2)} \approx \rho_0 \left(1 - \frac{15}{7\pi^2}\bar{m}^2 S^2\right), \tag{26}$$

where ρ_0 is the density the neutrino would have if it were massless. These expressions are correct to order $(\bar{m}S)^2$, though the expansion does not generalize straightforwardly to higher order. More generally one can find a relation between the multipoles of different velocity weight in the perturbed universe; using the definition given in Eq. (10) and expanding in $\bar{m}S$ we have

$$J^{(i+1)}_{A_l} \approx J^{(i)}_{A_l} - m^2 S^2 \frac{4\pi(-2)^l (l!)^2}{S^4 (2l+1)!} \int_0^\infty \mathrm{d}q \, \epsilon \, F_{A_l}. \tag{27}$$

To this order we can evaluate the last integral assuming the neutrinos are massless. Starting from a locally isotropic distribution, the momentum dependence of the F_{A_l} follows that of the $l = 1$ and 2 source terms in Eq. (21), so that

$$F_{A_l} \propto q \partial_q F \tag{28}$$

for $l > 0$. Using this and integrating by parts we have

$$J^{(i+1)}_{A_l} \approx J^{(i)}_{A_l} \left(1 - \frac{5}{7\pi^2}\bar{m}^2 S^2\right) \approx J^{(i)}_{A_l} \sqrt{\frac{3p}{\rho}} \tag{29}$$

for $l > 0$. In a similar manner, using $\mathcal{V}_a \propto q\partial_q F$ in the massless limit, we find

$$\chi^{(i+1)}_a \approx \chi^{(i)}_a \left(1 - \frac{5}{7\pi^2}\bar{m}^2 S^2\right) \approx \chi^{(i)}_a \sqrt{\frac{3p}{\rho}}. \tag{30}$$

To this order in $\bar{m}S$, Eqs. (29) and (30) provide closure conditions that allows reduction of the two dimensional momentum-integrated hierarchy, Eqs. (13) and (15), to a one-dimensional hierarchy for propagating the $J^{(0)}_{A_l}$ and χ_a that are required for coupling to other perturbations. The approximation is good for $(l + 2i)\bar{m}^2 S^2 \ll 1$.

The non-relativistic regime

When the bulk of the neutrinos have $q \ll m_\nu S$ the energy density and pressure evolve in the FRW limit as

$$\begin{aligned} \rho &\approx \frac{180\rho_0}{7\pi^4}\left(\zeta_3 \bar{m}S + \frac{15\zeta_5}{2\bar{m}S} - \frac{945\zeta_7}{16(\bar{m}S)^3} + \dots\right), \\ p &\approx \frac{900\rho_0}{7\pi^4}\left(\frac{\zeta_5}{\bar{m}S} - \frac{63}{4}\frac{\zeta_7}{(\bar{m}S)^3} + \dots\right). \end{aligned} \tag{31}$$

The value of $\bar{m}$ today $(S = 1)$ is related to the neutrino mass by

$$m_\nu \approx 1.68 \times 10^{-4} \bar{m} \, \mathrm{eV}. \tag{32}$$

For masses of order $0.2\,\mathrm{eV}$ or larger $\bar{m} \gtrsim 10^3$, and the mean speed today is less than about $10^{-3}c$. Even in the low mass case the neutrinos will therefore have been non-relativistic for a significant fraction of the universe's evolution.

Since the velocities are becoming small one can accurately truncate the momentum-integrated hierarchy, Eq. (13), at some fairly low velocity weight. However in order to do this one does need to know initial values in the non-relativistic era for those momentum-integrated multipoles that are retained after truncation. Generally, these cannot be obtained from the one-dimensional hierarchy that is integrated in the relativistic regime since the eras of applicability of the relativistic and non-relativistic approximations do not overlap. (However, a way of interpolating between these two regimes with sufficient accuracy for CMB computations is discussed below.) One must therefore integrate the distribution function directly, using equations (21) and (25), until one enters the non-relativistic regime, at which point one can switch over to a truncated momentum-integrated hierarchy.

The evolution of small-scale perturbations depends more critically on the higher velocity weight multipoles due to the derivative coupling in Eq. (13). Performing a mode expansion in wavenumber as in the appendix, the truncated hierarchy only becomes accurate at rather later times for small wavenumbers, as discussed in Sec. III.

An approximate scheme

The relic neutrinos themselves have very low energy and will remain unobservable for the foreseeable future. We are therefore not interested in the neutrino perturbations *per se*, just in their effect on the matter and photon perturbations. In this subsection we suggest an efficient approximate scheme, based on results obtained above, for propagating the massive neutrino perturbations into the non-relativistic regime which we have found to be sufficiently accurate for the computation of the CMB power spectra at the one percent level. This approximate scheme is inadequate for the matter power spectrum, which is most efficiently calculated using a momentum grid in the relativistic era matched onto the truncated momentum-integrated hierarchy when the neutrinos are non-relativistic.

If we use Eqs. (26) and (29), we can write the momentum-integrated hierarchy, Eqs. (13) and (15), for $i = 0$ during the relativistic era in the form

$$\dot{J}^{(0)}_{A_l} + H\left[(3+l) - r(l-1)\right] J^{(0)}_{A_l} - r\tfrac{l}{2l+1}\mathrm{D}_{\langle a_l} J^{(0)}_{A_{l-1}\rangle} + \mathrm{D}^a J^{(0)}_{aA_l} + \delta_{1l}\rho A_{a_1}(1+w) - \delta_{2l}\frac{2w\rho}{5}(5-3w)\sigma_{a_1 a_2} = 0, \tag{33}$$

$$\dot{\chi}_a + H(r+3)\chi_a + \mathrm{D}_a \mathrm{D}^b J^{(0)}_b + 3\rho(1+w)\dot{h}_a = 0, \tag{34}$$

where $w \equiv p/\rho$ and $r = (3w)^{1/2}$. These equations can be used to evolve the perturbations accurately while the majority of the particles are relativistic.

To motivate the approximate scheme, we note that massive neutrinos *perturbations* affect the CMB power spectra in the acoustic region mainly by the damping effect on the gravitational potential of their free-streaming on these scales [16]. (Other effects, including changes in the position of the acoustic peaks, arise mainly from the variation in the background equation of state.) The effect on the potential is insensitive to the late time evolution of small-scale neutrino perturbations, so these can be computed in a rather coarse manner at later times.

At late times the higher velocity weight multipoles fall off rapidly. Truncating the full momentum-integrated hierarchy at $n_* = 3$, one can show that $J^{(1)}_a \approx r J^{(0)}_a$ and $\chi^{(1)}_a \approx r\chi^{(0)}_a$ are solutions on large scales if $r \approx 5w$. Note that these conditions are frame-invariant to the order of our velocity-weight truncation, and that they give the expected scaling as the momenta are redshifted since $r \propto 1/S^2$. If we use $r = 5w$ in Eqs. (33) and (34), and drop terms with velocity weight $n > 3$, we obtain a one-dimensional hierarchy for χ_a and the $J^{(0)}_{A_l}$ with $l \leq 3$ that is consistent on large scales with the full two-dimensional hierarchy truncated at velocity weight 3. This suggests that if we evolve Eqs. (33) and (34) with r interpolating between $(3w)^{1/2}$ and $5w$, we obtain an approximate scheme that is accurate at early times and gives the correct late time behaviour for large-scale low-l multipoles. We found that using

$$r = \left(\frac{5}{3}\right)^{\bar{m}S/(\bar{m}S+200)} (3w)^{[\bar{m}S+2]/[\bar{m}S+4]} \tag{35}$$

works well and gives CMB power spectra accurate at the one percent level, as shown in Fig. 1. The evolution of the neutrino modes is accurate until $3w$ becomes significantly less than one, as shown in Fig. 2, and is also approximately correct at late times on large scales. Small scales evolve incorrectly at late times, but this has no observable consequence for the CMB.

The matter power spectrum is more sensitive to the late-time massive neutrino evolution than the CMB power spectra. On scales below the horizon size when the neutrinos become relativistic, free-streaming prevents them from clustering in the relativistic era and thus suppresses the growth of CDM perturbations. Once the neutrinos become non-relativistic they infall and cluster with the dominant CDM [17]. The approximate scheme described above is not sufficiently accurate at late times for computing the matter power spectrum, and the truncated momentum-integrated hierarchy should be propagated instead. However the late time matter power spectrum only affects the CMB via second order effects such as lensing, and the CMB temperature anisotropy therefore does not depend sensitively on its accuracy.

Numerical implementation

The covariant equations can be split into scalar, vector and tensor parts and then expanded in terms of the appropriate eigenfunctions of the co-moving Laplacian. This gives sets of equations for the harmonic coefficients that can be propagated numerically. For details of the scalar and tensor cases see the appendices.

The background evolution is affected by the neutrino density and pressure, which is a function only of $\bar{m}S$. The numerical integrations over the distribution function required to compute background quantities therefore only need be performed once for each value of $\bar{m}S$, and the values can be stored and re-used with different neutrino masses. Equations (26) and (31) can be used to calculate quickly the values at low and high values of $\bar{m}S$ respectively.

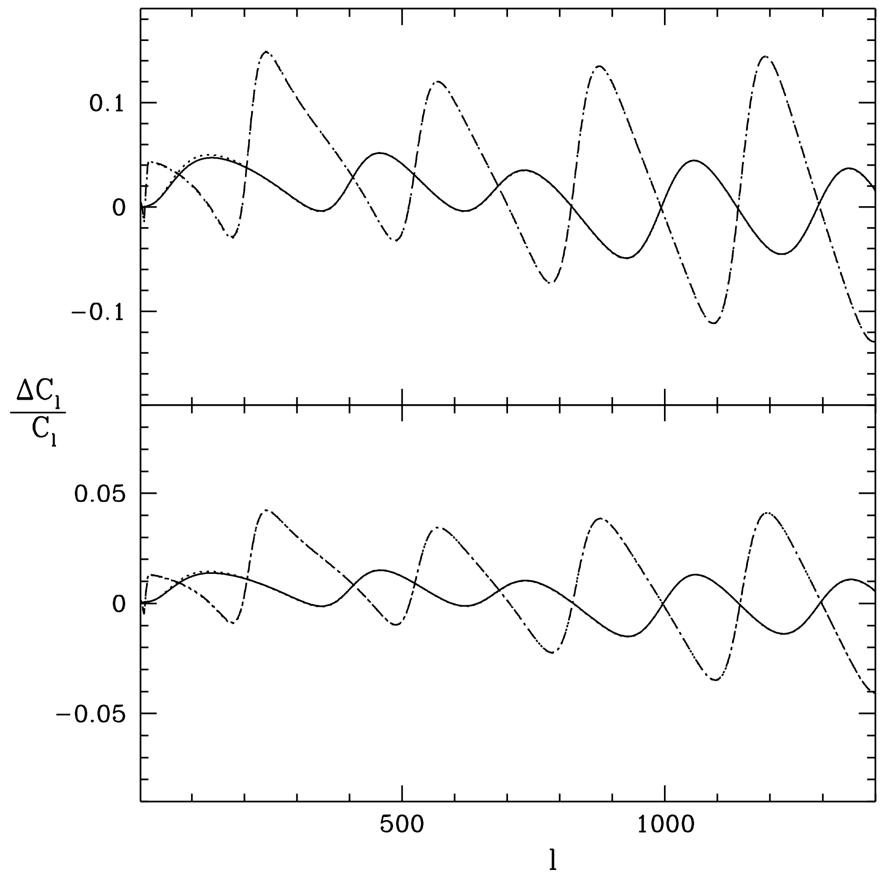

FIG. 1: The change in the scalar CMB temperature (solid lines) and E polarization (dashed line) power spectra compared to a CDM model, with $\Omega_\nu h^2 = 6.7 \times 10^{-3}$, $N_\nu = 3$ (top panel) and $\Omega_\nu h^2 = 2 \times 10^{-3}$, $N_\nu = 1$ (bottom panel) computed by integrating the distribution function perturbations. The dotted lines show the results from using the approximate Eq. (33), which agree very closely almost everywhere. We used $h = 0.69$, $\Omega_\Lambda = 0.7$, $\Omega_K = 0$, $\Omega_b h^2 = 0.022$.

For the computation of CMB anisotropies we require approximately one percent accuracy in the power spectrum for comparison with forthcoming accurate data. We implement the approximate scheme which is accurate at the one percent level in general, and much more so if the neutrinos are light. For one massive species this is about three times faster than propagating the distribution function directly, and the massive neutrino evolution no longer dominates the computation time.

For comparison, and for computing the matter power spectrum, we also propagate the multipoles of the distribution function directly. When the momentum has redshifted sufficiently that the neutrinos are non-relativistic we switch to propagating the truncated momentum-integrated hierarchy. We keep terms up to $n_* = 3$, giving four equations to propagate (the two highest order equations have analytic solutions $\propto S^{-3}$). The starting values for the momentum-integrated variables can be computed by integrating numerically over momentum at the switch-over point. Since the higher order variables are most important on small scales the switch-over point needs to be rather later for modes with large wavenumbers. This is a little slower than the approximate scheme, but can be made arbitrarily accurate by delaying the switch-over until the neutrinos have lower velocities.

Hot dark matter affects the large-scale tensor power spectrum predominantly through the change in the background equation of state. Tensor modes interact with the neutrinos only on sub-horizon scales on which they are decaying away, rapidly becoming negligible compared to the scalar perturbations.

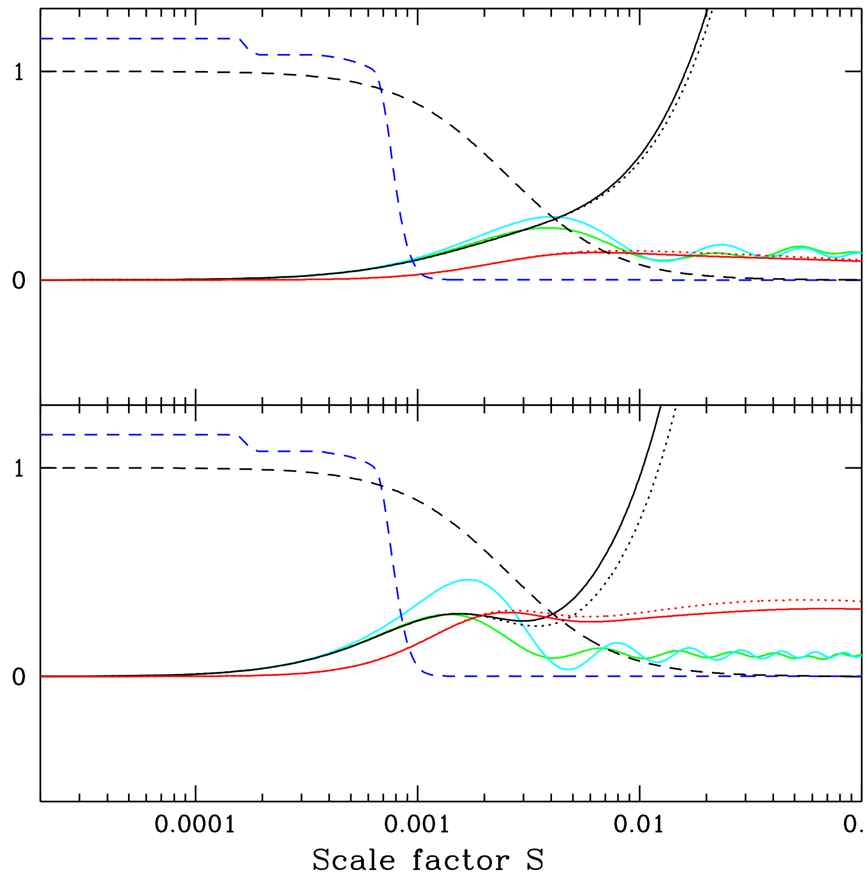

FIG. 2: The evolution of $k = 0.005\mathrm{Mpc}^{-1}$(top panel) and $k = 0.01\mathrm{Mpc}^{-1}$ (bottom panel) scalar modes, with three degenerate massive neutrinos giving $\Omega_\nu h^2 = 6.7 \times 10^{-3}$. The blue and black dashed lines respectively show the background ionization fraction and $3p_\nu/\rho_\nu$. The solid black and red lines show the massive neutrino density perturbation and momentum density, the dotted lines show the result of using Eq. (33) rather than integrating the distribution function. The green and cyan lines show the massless neutrino and photon perturbations. The perturbations are scaled corresponding to an initial curvature perturbation of 0.2 and are evaluated in the zero acceleration frame.

V. CONCLUSION

We have derived momentum-integrated multipole equations describing the propagation of dark matter with non-zero velocity dispersion. In the relativistic and non-relativistic regimes there are accurate approximations that can be used to evolve perturbations efficiently. We focussed on massive neutrino perturbations, though the same techniques could be used for matter with a different distribution function. For distributions with intermediate velocities one has to evolve the distribution function directly to get accurate results, though we found that for computing the CMB power spectrum a fast approximate scheme was sufficient to take account of the effect of massive neutrinos. We have implemented the scalar equations numerically and our fast parallelized CMB anisotropy code is publicly available at http://camb.info.

Acknowledgements

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APPENDIX A: THE SCALAR EQUATIONS

It is convenient to perform a harmonic expansion in terms of eigenfunctions Q^k of the co-moving Laplacian $S^2\mathrm{D}^a\mathrm{D}_a$,

$$S^2\mathrm{D}^a\mathrm{D}_a Q^k = k^2 Q^k. \tag{A1}$$

We define scalar coefficients for the harmonic expansion of PSTF tensors in terms of

$$Q^k_{A_l} \equiv (S/k)^l \mathrm{D}_{\langle a_1} \dots \mathrm{D}_{a_l\rangle} Q^k \tag{A2}$$

as follows:

$$J^{(i)}_{A_l} = \rho \sum_k I^{(i)}_l Q^k_{A_l}, \qquad \chi^{(i)}_a = \rho \sum_k k I^{(i)}_0 Q^k_a. \tag{A3}$$

We leave the k-dependence of the expansion coefficients implicit. The acceleration, shear and gradient of the scale factor can be expanded similarly as

$$A_a = \sum_k \tfrac{k}{S} A Q^k_a, \qquad \sigma_{ab} = \sum_k \tfrac{k}{S}\sigma Q^k_{ab}, \qquad h_a = \sum_k k h Q^k_a. \tag{A4}$$

Inserting the harmonic expansion into the multipole equations gives the propagation equations

$$\begin{aligned} I^{(i)\prime}_l \;&+\; \mathcal{H}\left[(1-n)I^{(i+1)}_l + (n-3w^{(1)})I^{(i)}_l\right] + k\left[\tfrac{l+1}{2l+1}\beta_{l+1} I^{(i)}_{l+1} - \tfrac{l}{2l+1} I^{(i+1)}_{l-1}\right] \\ &-\; \delta_{l2}\tfrac{2}{5}k\sigma\left[(3+n)w^{(i+1)} + (1-n)w^{(i+2)}\right] + \delta_{l1} kA\left[(2+n)w^{(i)} + (2-n)w^{(i+1)}\right] \\ &+\; \delta_{l0} 3h'\left[(1-n)w^{(i+1)} + (3+n)w^{(i)}\right] = 0, \end{aligned} \tag{A5}$$

where $3w^{(i)} \equiv J^{(i)}/\rho$, the dash denotes the derivate with respect to conformal time, $\mathcal{H} = S'/S$ is the conformal Hubble parameter, and $\beta_l \equiv 1 - (l^2-1)K/k^2$ where K is the curvature constant.

The scalar equations for the distribution function are obtained by expanding the multipoles in terms of scalar harmonics as

$$\mathcal{V}_a = F\sum_k k F_0 Q^k_a, \qquad F_{A_l} = F\frac{(2l+1)!}{(-2)^l (l!)^2}\sum_k F_l Q^k_{A_l}, \tag{A6}$$

giving the multipole equations

$$F'_l + k\tfrac{q}{\epsilon}\left(\tfrac{l+1}{2l+1}\beta_{l+1}F_{l+1} - \tfrac{l}{2l+1}F_{l-1}\right) + \left[\delta_{2l}\tfrac{2}{15}k\sigma - \delta_{1l}\tfrac{1}{3}k\left(\tfrac{\epsilon}{q}A + \tfrac{q}{\epsilon}h\right) - \delta_{0l}h'\right]\frac{\mathrm{d}\ln F}{\mathrm{d}\ln q} = 0. \tag{A7}$$

Choosing the frame u^a such that $A_a = 0$ these equations are equivalent to the synchronous gauge equations used by other codes [10, 12]. Assuming the neutrinos are initially highly relativistic the initial conditions for $F_l(q)$ are the same as for massless neutrinos:

$$F_l(q) = -\frac{I^{(0)}_l}{4}\frac{\mathrm{d}\ln F}{\mathrm{d}\ln q}. \tag{A8}$$

Later, when the particles are no longer highly relativistic, we can calculate the momentum-integrated multipoles by integrating over q

$$I^{(i)}_l = \frac{4\pi}{\rho S^4}\int_0^\infty \mathrm{d}q\, q^2 \epsilon\left(\frac{q}{\epsilon}\right)^{l+2i} F F_l. \tag{A9}$$

We choose to use the gauge in which $A_a = 0$, and keep terms up to $l+2i = n_* = 3$. This gives the four energy-integrated scalar equations

$$I^{(0)\prime}_0 + \mathcal{H}\left(I^{(1)}_0 - 3w^{(1)}I^{(0)}_0\right) + kI^{(0)}_1 + 3\left(1 + w^{(1)}\right)h' \;=\; 0 \tag{A10}$$

$$I^{(1)\prime}_0 + \mathcal{H}\left(2 - 3w^{(1)}\right)I^{(1)}_0 + kI^{(1)}_1 + 15w^{(1)}h' \;=\; 0 \tag{A11}$$

$$I^{(0)\prime}_1 + \mathcal{H}\left(1 - 3w^{(1)}\right)I^{(0)}_1 + \tfrac{1}{3}k\left(2\beta_2 I^{(0)}_2 - I^{(1)}_0\right) \;=\; 0 \tag{A12}$$

$$I^{(0)\prime}_2 + \mathcal{H}\left(2 - 3w^{(1)}\right)I^{(0)}_2 + \tfrac{1}{5}k\left(3\beta_3 I^{(0)}_3 - 2I^{(1)}_1\right) - 2kw^{(1)}\sigma \;=\; 0, \tag{A13}$$

where

$$I_3^{(0)} \propto \rho^{-1} S^{-6}, \qquad I_1^{(1)} \propto \rho^{-1} S^{-6}. \tag{A14}$$

The density perturbation and momentum density plotted in Fig. 2 correspond to the values of $I_0^{(0)}$ and $I_1^{(0)}$ in the $A_a = 0$ frame. In the non-covariant approach $I_0^{(0)}$ would just be the synchronous gauge $\delta\rho/\rho$.

APPENDIX B: THE TENSOR EQUATIONS

We follow the procedure in Ref. [18] and expand in terms of transverse tensor eigenfunctions Q^k_{ab} of the co-moving Laplacian,

$$S^2 \mathrm{D}^a \mathrm{D}_a Q^k_{ab} = k^2 Q^k_{ab}. \tag{B1}$$

We define scalar coefficients for the harmonic expansion of PSTF tensors with $l \geq 2$ in terms of

$$Q^k_{A_l} \equiv (S/k)^{(l-2)} \mathrm{D}_{\langle a_1} \ldots \mathrm{D}_{a_{l-2}} Q^k_{a_{l-1} a_l \rangle} \tag{B2}$$

as follows:

$$J^{(i)}_{A_l} = \rho \sum_k I_l^{(i)} Q^k_{A_l}, \qquad \sigma_{ab} = \sum_k \frac{k}{S} \sigma Q^k_{ab}. \tag{B3}$$

We leave the k-dependence of the expansion coefficients implicit.

Inserting the harmonic expansion into the multipole equations gives the $l \geq 2$ propagation equations

$$I_l^{(i)\prime} + \mathcal{H}\left[(1-n) I_l^{(i+1)} + (n - 3w^{(1)}) I_l^{(i)}\right] + k\left[\tfrac{(l+3)(l-1)}{(2l+1)(l+1)} \beta_{l+1} I_{l+1}^{(i)} - \tfrac{l}{2l+1} I_{l-1}^{(i+1)}\right] - \tfrac{2}{5} k\sigma \left[(3+n) w^{(i+1)} + (1-n) w^{(i+2)}\right] = 0, \tag{B4}$$

where $I_1^{(i)} = 0$ and $\beta_l \equiv 1 - (l^2 - 3)K/k^2$. The tensor equations for the propagation of the distribution function follow by expanding

$$F_{A_l} = F \frac{(2l+1)!}{(-2)^l (l!)^2} \sum_k F_l Q^k_{A_l}, \tag{B5}$$

giving the multipole equations

$$F_l' + k\tfrac{q}{\epsilon}\left[\tfrac{(l+3)(l-1)}{(2l+1)(l+1)} \beta_{l+1} F_{l+1} - \tfrac{l}{2l+1} F_{l-1}\right] + \tfrac{2}{15} k\sigma \frac{\mathrm{d}\ln F}{\mathrm{d}\ln q} = 0, \tag{B6}$$

where $F_1 = 0$.

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